

How to Lasso a Plane Gravitational Wave

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Abstract. Beginning with the stress-energy tensor of an elastic string this paper derives a relativistic string and its form in a parallel transported Fermi frame including its reduction in the Newtonian limit to a Cosserat string. In a Fermi frame gravitational curvature is seen to induce three dominant relative acceleration terms dependent on: position, velocity and position, strain and position, respectively. An example of a string arranged in an axially flowing ring (a lasso) is shown to have a set of natural frequencies that can be parametrically excited by a monochromatic plane gravitational wave. The lasso also exhibits, in common with spinning particles, oscillation about geodesic motion in proportion to spin magnitude and wave amplitude when the spin axis lies in the gravitational wave front. Coordinate free notation is used throughout including the development of the properties of the Fermi frame.

1. Introduction

The use of an elastic continuum to detect gravitational effects is not new [1], but a space based gravitational wave detector would have to be of considerable size to resonate with the low frequency and strength of waves theoretically predicted [2]. The use of a one dimensional elastic continuum holds out the possibility of producing such a resonant detector without the burden of high payload for shipment to space. A one dimensional elastic continuum is known as a Cosserat string and is the limit of a Cosserat rod as the cross sectional area tends to zero. The theory can be consulted in [3] and is fundamentally formulated in the Lagrangian picture in which each element of the string is labeled by $s \in [0, L_0]$ where L_0 is the string's reference (unstretched) length. The evolution of the string at time t is given by a space curve

$$s \in [0, L_0] \mapsto \mathbf{R}(s, t)$$

and obeys the Cosserat string equation

$$\rho A \partial_t^2 \mathbf{R} = \partial_s \mathbf{N} + \mathbf{F} \tag{1}$$

where in general the mass density

$$s \in [0, L_0] \mapsto \rho(s)$$

and area

$$A \in [0, L_0] \mapsto A(s)$$

The mass density, ρ , and A are assumed constant in this paper allowing A to be removed and ρ to be reinterpreted as mass per unit length. To close the equations (1) the contact force, \mathbf{N} and external force \mathbf{F} are prescribed. In the string model the contact force \mathbf{N} is aligned along the tangent to the space-curve. A simple constitutive relation due to Kirchhoff is given by

$$\mathbf{N} \equiv EA(|\mathbf{R}'| - 1) \frac{\mathbf{R}'}{|\mathbf{R}'|} \quad (2)$$

where E is Young's modulus. The Kirchhoff constitutive relation is linear and is a good model for small strains for strings in permanent tension. Generalizations are given in [3] and an application for rods in [4].

The dynamics of a Cosserat string in a Newtonian spherically symmetric gravitational field is modeled by setting

$$\mathbf{F} \equiv -\frac{GM\rho A\mathbf{R}}{|\mathbf{R}|^3}.$$

To model the stresses on a Cosserat string due to spacetime curvature, for example, in a gravitational wave, or in orbit about a black hole, one can supply a 'tidal' force lifted from general relativity:

$$\mathbf{F} \equiv \rho A \text{Riem}(\dot{C}, \mathbf{R}, \dot{C}, -),$$

where \dot{C} is the tangent to the world line of a comoving observer and $Riem$ is the curvature tensor [5]. The tidal term represents the relative acceleration of two *non interacting* neighbouring particles and it is not clear how this term should be modified when particles are not in free fall and are interacting. This subject is addressed in this paper.

2. The relativistic string

2.1. Stress-energy tensor for a string

For a general spacetime \mathcal{M} with metric g , the world sheet of a string can be represented by two orthonormal vector fields V and W such that $g(V, V) = -1$, $g(W, W) = 1$ and $g(V, W) = 0$. We associate the integral curves of V with material points on the string. The $(2, 0)$ stress-energy tensor for the string can be written

$$T \equiv \rho \tilde{V} \otimes \tilde{V} + p \tilde{W} \otimes \tilde{W}$$

where the metric dual $\tilde{V} \equiv g(V, -)$, ρ is the density and p is the pressure at points on the string. For a string it is convenient to interpret ρ as mass per unit length and p as force. To find the equation for a relativistic string we wish to set the divergence $\nabla \cdot T \equiv (\nabla_{X_a} T)(-, X^a)$ to zero, where ∇ is compatible with the metric g in the absence of the string and $\{X_a\}$ is some basis. However, $\nabla \cdot T$ is not defined in directions out of the tangent plane of the string world sheet. To this end denoting an orthonormal frame $\{X_a\}$, $a \in [0, 3]$ such that $X_0 = V$, $X_1 = W$, we demand $\nabla_{X_2} T = \nabla_{X_3} T = 0$. Then

$$0 = \nabla \cdot T \equiv (\nabla_{X_a} T)(-, X^a)$$

$$\begin{aligned}
&= -\nabla_V T(-, V) + \nabla_W T(-, W) \\
&= \tilde{V} \nabla_V \rho + \rho \tilde{V} \nabla \cdot V + \rho \nabla_V \tilde{V} + p \nabla_W \tilde{W} + \tilde{W} \nabla_W p + p \tilde{W} \nabla \cdot W \quad (3)
\end{aligned}$$

where we use $\tilde{V}(\nabla_V V) = \tilde{W}(\nabla_W W) = 0$ to eliminate two terms. Also note that $\nabla \cdot V = \tilde{W}(\nabla_W V) = -\tilde{V}(\nabla_W W)$ and $\nabla \cdot W = -\tilde{V}(\nabla_V W) = \tilde{W}(\nabla_V V)$. Taking the metric dual of (3) gives the relativistic string equation

$$V \nabla_V \rho + \rho V \nabla \cdot V + \rho \nabla_V V + p \nabla_W W + W \nabla_W p + p W \nabla \cdot W = 0. \quad (4)$$

3. The Special relativistic Cosserat string

For a general spacetime \mathcal{M} with metric g , a string can be defined by the embedding R

$$\begin{aligned}
&\mathbb{R} \times [0, 1] \rightarrow \mathcal{M}, \\
&(\tau, \sigma) \mapsto R(\tau, \sigma).
\end{aligned}$$

Now let (\mathcal{M}, g) be Minkowski spacetime with natural Cartesian coordinates $\{t, x, y, z\}$ and metric

$$g \equiv -dt \otimes dt + dx \otimes dx + dy \otimes dy + dz \otimes dz$$

then the string can be defined by the mapping

$$\begin{aligned}
&\mathbb{R} \times [0, 1] \rightarrow \mathcal{M}, \\
&(\tau, \sigma) \mapsto R(\tau, \sigma) = [t = T(\tau, \sigma), \mathbf{R}(\tau, \sigma)] \\
&\quad \equiv [t = T(\tau, \sigma), x = X(\tau, \sigma), y = Y(\tau, \sigma), z = Z(\tau, \sigma)],
\end{aligned}$$

Using notation $\dot{(\cdot)} \equiv \partial/\partial\tau$ and $(\cdot)' \equiv \partial/\partial\sigma$ we can write

$$\partial_\tau = \dot{T} \partial_t + \dot{\mathbf{R}} \quad (5)$$

and

$$\partial_\sigma = T' \partial_t + \mathbf{R}' \quad (6)$$

where

$$\dot{\mathbf{R}} \equiv \dot{X} \partial_x + \dot{Y} \partial_y + \dot{Z} \partial_z$$

$$\mathbf{R}' \equiv X' \partial_x + Y' \partial_y + Z' \partial_z.$$

Then setting

$$V = \alpha(\dot{T} \partial_t + \dot{\mathbf{R}})$$

gives $g(V, V) = -1$, with α defined

$$\alpha \equiv [-g(\partial_\tau, \partial_\tau)]^{-1/2} = [\dot{T}^2 - \dot{\mathbf{R}} \cdot \dot{\mathbf{R}}]^{-1/2}$$

and $\dot{T}^2 > \dot{\mathbf{R}} \cdot \dot{\mathbf{R}}$ for V to be time like. Similarly

$$W = \beta(T' \partial_t + \mathbf{R}')$$

gives $g(W, W) = 1$, with β defined

$$\beta \equiv [g(\partial_\sigma, \partial_\sigma)]^{-1/2} = [\mathbf{R}' \cdot \mathbf{R}' - T'^2]^{-1/2}$$

and we demand $\mathbf{R}' \cdot \mathbf{R}' > T'^2$. We now have

$$V \nabla_V \rho = \alpha^2 \dot{\rho} (\dot{T} \partial_t + \dot{\mathbf{R}}), \quad (7)$$

$$W \nabla_W p = \beta^2 p' (T' \partial_t + \mathbf{R}'), \quad (8)$$

$$\rho \nabla_V V = \rho \alpha [(\dot{\alpha} \dot{T} + \alpha \ddot{T}) \partial_t + \dot{\alpha} \dot{\mathbf{R}} + \alpha \ddot{\mathbf{R}}], \quad (9)$$

$$p \nabla_W W = p \beta [(\beta' T' + \beta T'') \partial_t + \beta' \mathbf{R}' + \beta \mathbf{R}'']. \quad (10)$$

$$\begin{aligned} \rho V \nabla \cdot V &= \rho V [g(W, \nabla_W V) - g(V, \nabla_V V)] \\ &= \rho V g(W, \nabla_W V) \\ &= -\rho V g(V, \nabla_W W) \\ &= \rho \alpha^2 \beta^2 (T'' \dot{T} - \mathbf{R}'' \cdot \dot{\mathbf{R}}) (\dot{T} \partial_t + \dot{\mathbf{R}}), \end{aligned} \quad (11)$$

$$\begin{aligned} p W \nabla \cdot W &= p W [-g(V, \nabla_V W) + g(W, \nabla_W W)] \\ &= -p W g(V, \nabla_V W) \\ &= p W g(W, \nabla_V V) \\ &= p \alpha^2 \beta^2 (\ddot{\mathbf{R}} \cdot \mathbf{R}' - \ddot{T} T') (T' \partial_t + \mathbf{R}'), \end{aligned} \quad (12)$$

Where the orthonormality condition, $g(V, W) = 0 = g(\partial_\tau, \partial_\sigma)$, implying

$$\dot{T} T' = \dot{\mathbf{R}} \mathbf{R}', \quad (13)$$

has been used to simplify equations (11) and (12). Further simplification comes from noting that, (13) also implies

$$(\dot{T} T')' = \dot{T}' T' + \dot{T} T'' = \dot{\mathbf{R}}' \cdot \mathbf{R}' + \dot{\mathbf{R}} \cdot \mathbf{R}''$$

so that

$$\begin{aligned} T'' \dot{T} - \mathbf{R}'' \cdot \dot{\mathbf{R}} &= \dot{\mathbf{R}}' \cdot \mathbf{R}' - \dot{T}' T' \\ &= \frac{1}{2} \frac{\partial}{\partial \tau} (\mathbf{R}' \cdot \mathbf{R}' - (T')^2) \\ &= \frac{1}{2} \frac{\partial}{\partial \tau} \left(\frac{1}{\beta^2} \right) = -\frac{\dot{\beta}}{\beta^3} \end{aligned}$$

Defining $\rho_0 \equiv \rho/\beta$ and adding equations (7), (11) and (9) yields

$$V \nabla_V \rho + \rho V \nabla \cdot V + \rho \nabla_V V = \alpha \beta [\partial_\tau (\alpha \dot{T} \rho_0) \partial_t + \partial_\tau (\alpha \rho_0 \dot{\mathbf{R}})]. \quad (14)$$

Similarly

$$\frac{\partial}{\partial \tau} (\dot{T} T') = \ddot{T} T' + \dot{T} \dot{T}' = \ddot{\mathbf{R}} \cdot \mathbf{R}' + \dot{\mathbf{R}} \cdot \dot{\mathbf{R}}'$$

implies

$$\begin{aligned} \ddot{\mathbf{R}} \cdot \mathbf{R}' - \ddot{T} T' &= \dot{T} \dot{T}' - \dot{\mathbf{R}}' \cdot \dot{\mathbf{R}} \\ &= \frac{1}{2} (\dot{T}^2 - \dot{\mathbf{R}} \cdot \dot{\mathbf{R}})' \\ &= \frac{1}{2} \left(\frac{1}{\alpha^2} \right)' = -\frac{\alpha'}{\alpha^3}. \end{aligned}$$

Then a similar simplification can be made by defining $p_0 \equiv p/\alpha$, which gives on adding (8),(12) and (10)

$$W\nabla_W p + pW\nabla \cdot W + p\nabla_W W = \alpha\beta \left[\left(\frac{\beta p_0}{\dot{T}} \dot{\mathbf{R}} \cdot \mathbf{R}' \right)' \partial_t + (\beta p_0 \mathbf{R}')' \right] \quad (15)$$

Equations (14) and (15) imply both

$$\frac{\partial}{\partial \tau} (\alpha \dot{T} \rho_0) + \left(\frac{\beta p_0}{\dot{T}} \dot{\mathbf{R}} \cdot \mathbf{R}' \right)' = 0 \quad (16)$$

and

$$\frac{\partial}{\partial \tau} (\alpha \rho_0 \dot{\mathbf{R}}) + (\beta p_0 \mathbf{R}')' = 0. \quad (17)$$

Note, the familiar quantities: velocity, $\mathbf{v} \equiv \dot{\mathbf{R}}/\dot{T}$; speed $v \equiv \sqrt{\mathbf{v} \cdot \mathbf{v}}$ and Lorentz factor $\gamma \equiv (1 - v^2)^{-1/2} = \dot{T}\alpha$, can be used to aid physical interpretation and clean up the appearance of (16) and (17) still further. To complete the relativistic string in Minkowski spacetime there is freedom to relate T on a world line corresponding to one material point to the proper time of some observer, e.g. we could choose $t = T(\tau, \sigma_0) = \tau$ for some $\sigma_0 \in [0, 1]$, implying $\dot{T}(\tau, \sigma_0) = 1$. This choice guarantees for general points at non-relativistic speeds, $v \rightarrow 0$, that $t = T(\tau, \sigma) \rightarrow \tau$, $\alpha \rightarrow 1$ and $\beta \rightarrow 1/|\mathbf{R}'|$.

Given constitutive relations for ρ_0 and p_0 the special relativistic equations (16) and (17) are 4 equations for the unknowns $T(\tau, \sigma)$ and $\mathbf{R}(\tau, \sigma)$. Setting ρ_0 to be independent of τ may be physically interpreted as the string having no fluid properties, i.e. purely elastic. However, in the relativistic context here, ρ_0 includes energy density with a time dependent contribution, for example, due to heating and elastic potential. Choosing ρ_0 independent of τ is necessary to the reduction in the Newtonian limit to the Cosserat string. With this choice (17) becomes in the Newtonian limit:

$$\rho_0 \ddot{\mathbf{R}} = \mathbf{N}' \quad (18)$$

where $\mathbf{N} \equiv -\beta p_0 \mathbf{R}' \rightarrow -p_0 \mathbf{R}'/|\mathbf{R}'|$. Equation (18) is the Cosserat string (1) in the absence of external forces with the notation reference mass per unit length, ρ_0 , replacing ρA , the reference mass per unit volume times cross-sectional area. We note that (16) may be written

$$\partial_\tau (\gamma \rho_0) = (\mathbf{v} \cdot \mathbf{N})'$$

which relates the rate of change of energy density to work density.

4. Fermi normal coordinates

This section takes a geometric approach to the development of Fermi coordinates for use in discussing the dynamics of the string in a gravitational field as seen by a geodesic observer. For a standard development of Fermi normal coordinates see [6]. A Fermi normal frame represents the coordinates associated with the one-one exponential map between an observer's tangent plane and the manifold in the observer's local neighbourhood. An event local to the observer is given a time component equal to

the proper time of the observer and three Cartesian like coordinates give the position. Let $\{x^a\}, a \in [0, 3]$ denote these coordinates, where coordinate, x^0 is the proper time of the observer. Henceforth only geodesic observers will be considered. For an observer curve, C , a Fermi frame, $X_a \equiv \partial_{x^a}, a \in [0, 3]$, satisfies orthonormality

$$g(X_a, X_b)|_C = \eta_{ab} \quad (19)$$

and parallel transport

$$\nabla_{X_a} X_b|_C = 0. \quad (20)$$

The C subscript indicating evaluation at points on the curve - the *origin* of the spatial coordinates. Equation (20) follows from orthonormality (19) and $\mathcal{L}_{X_a} X_b = \nabla_{X_a} X_b - \nabla_{X_b} X_a = 0$, since on the curve

$$\begin{aligned} g(\nabla_{X_a} X_b, X_c) &= -g(X_b, \nabla_{X_a} X_c) \\ &= -g(X_b, \nabla_{X_c} X_a) \\ &= g(X_a, \nabla_{X_c} X_b) \\ &= g(X_a, \nabla_{X_b} X_c) \\ &= -g(X_c, \nabla_{X_b} X_a) \\ &= -g(X_c, \nabla_{X_a} X_b) \end{aligned}$$

implies $g(\nabla_{X_a} X_b, X_c) = 0$ for arbitrary $a, b, c \in [0, 3]$.

4.1. The evaluation of the connection near the world line

Any event near the world line of an observer not only defines its Fermi coordinates but also a unique spacelike geodesic connecting the observer with the event. This allows tensors to be parallel transported to the world line in a special way and Taylor expansion of tensor fields in the neighbourhood of C is well defined. A tensor $A|_U$ at spatial position corresponding to the spacelike vector U via the exponential map, can be expanded

$$A|_U = A|_C + \nabla_U A|_C + \frac{1}{2} \nabla_U \nabla_U A|_C + O(|U|^3)$$

where $A|_C$ indicates its value on the world line. At arbitrary time x^0 an arbitrary spacelike tangent vector, U , at $C(x^0)$ is mapped by the exponential map to a unique spacelike geodesic \mathcal{U} . Let \mathcal{U} have affine parameter u so that $U = \partial_u$. Then $\nabla_U U = 0$ along \mathcal{U} implies that $\nabla_U \nabla_U U = 0$ along \mathcal{U} . Let V be another arbitrary tangent vector at $C(x^0)$ then using $\nabla_U \nabla_U U = \nabla_V \nabla_V V = 0$ implies on setting $Y = U + V$ that

$$\nabla_Y \nabla_Y Y = 0 = \nabla_U \nabla_U V + \nabla_U \nabla_V V + \nabla_V \nabla_U U + \nabla_V \nabla_V U$$

We also find that

$$\begin{aligned} \nabla_Y Y = 0 &= \nabla_U U + \nabla_U V + \nabla_V U + \nabla_V V \\ &= 2\nabla_U V \end{aligned}$$

confirming (20). Using the above result, let $Z = U + V + W$ where U, V, W are all arbitrary spacelike vectors in the tangent space of $C(x^0)$ then

$$\nabla_Z \nabla_Z Z = 0 = 3(\nabla_U \nabla_V W + \nabla_V \nabla_W U + \nabla_W \nabla_U V).$$

Using this result and the definition of the Riemann curvature tensor in terms of arbitrary vector fields E, F, G ,

$$\text{Riem}(E, F, G, -) \equiv \nabla_E \nabla_F G - \nabla_F \nabla_E G - \nabla_{[E, F]} G$$

where the last term vanishes when $[E, F] \equiv \mathcal{L}_E F = 0$ as is the case here, implies

$$\begin{aligned} & \text{Riem}(U, V, W, -) + \text{Riem}(U, W, V, -) \\ &= \nabla_U \nabla_V W - \nabla_V \nabla_U W + \nabla_U \nabla_W V - \nabla_W \nabla_U V \\ &= 3\nabla_U \nabla_V W - \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_W \nabla_U V \\ &= 3\nabla_U \nabla_V W \end{aligned}$$

so in particular for $a, b \in [1, 3]$ and U spacelike

$$\nabla_U \nabla_{X_a} X_b|_C = \frac{1}{3}(\text{Riem}(U, X_a, X_b, -)|_C + \text{Riem}(U, X_b, X_a, -)|_C).$$

For the remaining cases, let $a \in [0, 3]$ then

$$\begin{aligned} \nabla_{X_0} X_a|_U &= \nabla_{X_a} X_0|_U = \nabla_U \nabla_{X_0} X_a|_C \\ &= \text{Riem}(U, X_0, X_a, -)|_C + \nabla_{X_0} \nabla_U X_a|_C \\ &= \text{Riem}(U, X_0, X_a, -)|_C. \end{aligned} \tag{21}$$

In particular for $a = 0$

$$\begin{aligned} \nabla_{X_0} X_0|_U &= \text{Riem}(U, X_0, X_0, -)|_C \\ &= -\text{Riem}(X_0, U, X_0, -)|_C. \end{aligned} \tag{22}$$

It is instructive to compare (22) with the equation of geodesic deviation. Let Q be a geodesic vector field so that $\nabla_Q Q = 0$. Introduce the field S orthogonal to Q such that $\mathcal{L}_S Q = 0$, then

$$\begin{aligned} \nabla_S \nabla_Q Q &= 0 = \nabla_S \nabla_Q Q - \nabla_Q \nabla_S Q + \nabla_Q \nabla_S Q \\ &= -\text{Riem}(Q, S, Q, -) + \nabla_Q \nabla_S Q \end{aligned}$$

so that using $\mathcal{L}_S Q = 0$ gives

$$\nabla_Q \nabla_Q S = \text{Riem}(Q, S, Q, -).$$

Now S represents the physical separation of two neighbouring geodesics. On the other hand U in (22) represents an observer a fixed displacement from a geodesic and so must be accelerating in an opposite sense to S to maintain its position.

We require one further property regarding the orthonormality of the Fermi frame. To $O(|U|)$:

$$\begin{aligned} g(X_a, X_b)|_U &= g(X_a, X_b)|_C + g(\nabla_U X_a, X_b)|_C + g(X_a, \nabla_U X_b)|_C \\ &= g(X_a, X_b)|_C. \end{aligned}$$

5. A relativistic string in Fermi normal coordinates

The above results suggest the definition of a new tensor associated with the Fermi frame. Let general vectors A, B be decomposed into the time components \mathcal{A}, \mathcal{B} and 3 space components \mathbf{A}, \mathbf{B} of the Fermi frame (e.g. $\mathcal{A} = A^0 X_0 \equiv A^0 \dot{C}$, $\mathbf{A} = A^a X_a$), then for spacelike displacement \mathbf{R} define the (2,1) *Fermi* tensor, $F_{\mathbf{R}}$ by

$$F_{\mathbf{R}}(A, B, -) \equiv \text{Riem}(\mathbf{R}, \mathcal{A}, \mathcal{B}, -) + \text{Riem}(\mathbf{R}, \mathcal{A}, \mathbf{B}, -) + \text{Riem}(\mathbf{R}, \mathbf{A}, \mathcal{B}, -) \\ + \frac{1}{3}[\text{Riem}(\mathbf{R}, \mathbf{A}, \mathbf{B}, -) + \text{Riem}(\mathbf{R}, \mathbf{B}, \mathbf{A}, -)]$$

For example, with $a \in [0, 3]$, the following term taken from (4) expands as

$$\rho \nabla_V V|_{\mathbf{R}} = \rho \alpha \dot{V}^a X_a + \rho F_{\mathbf{R}}(V, V, -),$$

where the first term is identical to that derived for flat spacetime and the second term contains the curvature. The definition of $F_{\mathbf{R}}$ yields

$$F_{\mathbf{R}}(V, V, -) = (\alpha \dot{T})^2 \text{Riem}(\mathbf{R}, \dot{C}, \dot{C}, -) + 2\alpha^2 \dot{T} \text{Riem}(\mathbf{R}, \dot{C}, \dot{\mathbf{R}}, -) \\ + \frac{2}{3}\alpha^2 \text{Riem}(\mathbf{R}, \dot{\mathbf{R}}, \dot{\mathbf{R}}, -).$$

The Fermi tensor allows a succinct upgrade of the special relativistic equations to Fermi relativistic equations. Using the general relativistic equation (4), the flat spacetime results in (14) and (15) and the definition of $F_{\mathbf{R}}$ we obtain:

$$0 = \alpha\beta[\partial_\tau(\alpha\dot{T}\rho_0)\partial_t + \partial_\tau(\alpha\rho_0\dot{\mathbf{R}})] + \alpha\beta\left[\left(\frac{\beta p_0}{\dot{T}}\dot{\mathbf{R}} \cdot \mathbf{R}'\right)' \partial_t + (\beta p_0 \mathbf{R}')'\right] \\ + \rho F_{\mathbf{R}}(V, V, -) + p F_{\mathbf{R}}(W, W, -) + \rho V F_{\mathbf{R}}(W, V, \tilde{W}) + p W F_{\mathbf{R}}(V, V, \tilde{W}), \quad (23)$$

which together with the substitutions: $V = \alpha(\dot{T}\dot{C} + \dot{\mathbf{R}})$, $W = \beta(T'\dot{C} + \mathbf{R}')$, $T' = \dot{\mathbf{R}} \cdot \mathbf{R}'/\dot{T}$, $\rho = \rho_0\beta$ and $p = p_0\alpha$, complete the Fermi relativistic string.

We note that at non relativistic speeds the dominant curvature term in $F_{\mathbf{R}}(V, V, -)$ is $\text{Riem}(\mathbf{R}, \dot{C}, \dot{C}, -)$. It can be shown that this is the dominant term in (23) and the string equation becomes in the Newtonian limit

$$\rho_0 \ddot{\mathbf{R}} = \mathbf{N}' + \rho_0 \text{Riem}(\dot{C}, \mathbf{R}, \dot{C}, -).$$

using antisymmetry of *Riem* in the first two slots. To order the terms in magnitude note that p_0/ρ_0 has dimensions Force/(mass per unit length), or speed squared and may be treated as $p_0/\rho_0 \sim c_s^2$ where c_s is the characteristic speed of ‘sound’ of the string. This may be taken in many cases to be greater than v avoiding ‘shocks’. Coupling the next two most dominant curvature terms to the Cosserat string then yields

$$\rho_0 \ddot{\mathbf{R}} - \mathbf{N}' = \rho_0 \text{Riem}(\dot{C}, \mathbf{R}, \dot{C}, -) \\ + \Pi_C \left[2\rho_0 \text{Riem}(\dot{C}, \mathbf{R}, \dot{\mathbf{R}}, -) + \frac{2}{3} \text{Riem}(\mathbf{R}, \mathbf{R}', \mathbf{N}, -) \right] \quad (24)$$

where Π_C is the projection onto the rest space of the observer.

Equation (24) has dimensions $ML^{-1}T^{-2}$. Using the constitutive relation (2) and

$$\mathbf{r} \equiv \frac{\mathbf{R}}{L_0}, \quad c_s \equiv \sqrt{\frac{E}{\rho_0}}, \quad \mathbf{n} \equiv (\nu - 1) \frac{\mathbf{r}'}{\nu},$$

(24) can be written

$$\ddot{\mathbf{r}} - c_s^2 \mathbf{n}' = \text{Riem}(\dot{C}, \mathbf{r}, \dot{C}, -) + \Pi_C \left[2\text{Riem}(\dot{C}, \mathbf{r}, \dot{\mathbf{r}}, -) + \frac{2c_s^2}{3} \text{Riem}(\mathbf{r}, \mathbf{r}', \mathbf{n}, -) \right].$$

Setting $c_s = 1$ gives

$$\ddot{\mathbf{r}} - \mathbf{n}' = \mathbf{f} \tag{25}$$

where

$$\mathbf{f} \equiv \text{Riem}(\dot{C}, \mathbf{r}, \dot{C}, -) + \Pi_C \left[2\text{Riem}(\dot{C}, \mathbf{r}, \dot{\mathbf{r}}, -) + \frac{2}{3} \text{Riem}(\mathbf{r}, \mathbf{r}', \mathbf{n}, -) \right]. \tag{26}$$

This fixes the unit of length to that of the unstretched string, L_0 ; the unit of speed to c_s ; and the unit of time to L_0/c_s .

In (25) the tidal force, $\text{Riem}(\dot{C}, \mathbf{r}, \dot{C}, -)$, is clearly the dominant force. However, the other terms are not completely negligible. Consider for example a string arranged in a closed circular ring spinning about a normal axis through its centre. Then this arrangement can respond so as to divert the ring from a geodesic path via the second and third terms (the tidal term cannot do this). There is a connection here with Dixon's equations [7] (see also [8] for the dynamics of spinning particles in gravitational waves) as well as an analogy with electromagnetic effects [9]. The first term is analogous to an electric field; the second term a magnetic field due to the coupling between mass current and the field. The third term, however, has no electromagnetic analogue and is generally weaker than the magnetic term. The next two sections investigate the dynamics of Cosserat string arranged in an axially flowing ring- a lasso.

6. Dynamics of a Cosserat string

This section seeks solutions to

$$\ddot{\mathbf{r}} = \mathbf{n}' + \mathbf{f} \tag{27}$$

where

$$\mathbf{n} = (|\mathbf{r}'| - 1) \frac{\mathbf{r}'}{|\mathbf{r}'|} \tag{28}$$

and $(\dot{}) \equiv \partial_\tau, ()' \equiv \partial_\sigma$. In the absence of an external force, $\mathbf{f} = 0$ a circular axially flowing solution given in a Cartesian frame $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is

$$\mathbf{r}_0(\sigma, \tau) = \frac{\nu}{2\pi} \cos(2\pi(\sigma + \mu\tau)) \mathbf{i} + \frac{\nu}{2\pi} \sin(2\pi(\sigma + \mu\tau)) \mathbf{j} \tag{29}$$

μ is the rotation frequency related to the constant strain, $\nu \equiv |\mathbf{r}'_0|$ by

$$\mu \equiv \sqrt{\frac{\nu - 1}{\nu}} \tag{30}$$

Defining an orthonormal frame rotating with each material point of the string by

$$\begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \mathbf{U}_3 \end{bmatrix} \equiv \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix} \quad (31)$$

where

$$\theta \equiv 2\pi(\sigma + \mu\tau) \quad (32)$$

solution (29) can be written

$$\mathbf{r}_0(\sigma, \tau) = \frac{\nu}{2\pi} \mathbf{U}_1. \quad (33)$$

Perturbing about this result by writing for some small parameter, ϵ

$$\mathbf{r} = \mathbf{r}_0 + \epsilon \mathbf{r}_1 + O(\epsilon^2) \quad (34)$$

and demanding that \mathbf{f} is

$$\mathbf{f} = O(\epsilon) \quad (35)$$

then inserting into (27) and (28) gives to $O(\epsilon)$ the equations for \mathbf{r}_1

$$\begin{aligned} & \left(\ddot{\eta} - \mu^2 \eta'' + 2\pi(1 + \mu^2) \xi' - 4\pi\mu \dot{\xi} + 4\pi^2(1 - \mu^2) \eta \right) \mathbf{U}_1 \\ & + \left(\ddot{\xi} - \xi'' - 2\pi(1 + \mu^2) \eta' + 4\pi\mu \dot{\eta} \right) \mathbf{U}_2 + (\ddot{\zeta} - \mu^2 \zeta'') \mathbf{U}_3 = \mathbf{f} \end{aligned} \quad (36)$$

where

$$\mathbf{r}_1 = \eta \mathbf{U}_1 + \xi \mathbf{U}_2 + \zeta \mathbf{U}_3 \quad (37)$$

The solution to the homogeneous equation, with $\mathbf{f} = 0$ is

$$\begin{aligned} \mathbf{r}_1 = \begin{bmatrix} \eta \\ \xi \\ \zeta \end{bmatrix} = & \begin{bmatrix} A \cos(\omega_0 \tau) + B \sin(\omega_0 \tau) + C \\ \frac{4\pi\mu}{\omega_0} (-A \sin(\omega_0 \tau) + B \cos(\omega_0 \tau)) + \pi \frac{1-\mu_0^2}{\mu_0} C \tau + D \\ 0 \end{bmatrix} \\ & + \sum_{J=1}^2 \begin{bmatrix} a_{J1} \alpha_{J1} \cos(\omega_{J1} \tau + 2\pi\sigma) + b_{J1} \alpha_{J1} \sin(\omega_{J1} \tau + 2\pi\sigma) \\ a_{J1} \sin(\omega_{J1} \tau + 2\pi\sigma) - b_{J1} \cos(\omega_{J1} \tau + 2\pi\sigma) \\ 0 \end{bmatrix} \\ & + \sum_{k=2}^{\infty} \sum_{J=1}^4 \begin{bmatrix} a_{Jk} \alpha_{Jk} \cos(\omega_{Jk} \tau + 2\pi k \sigma) + b_{Jk} \alpha_{Jk} \sin(\omega_{Jk} \tau + 2\pi k \sigma) \\ a_{Jk} \sin(\omega_{Jk} \tau + 2\pi k \sigma) - b_{Jk} \cos(\omega_{Jk} \tau + 2\pi k \sigma) \\ 0 \end{bmatrix} \\ & + \sum_{k=0}^{\infty} \begin{bmatrix} 0 \\ 0 \\ c_k \cos(2\pi k(\mu\tau + \sigma)) + d_k \sin(2\pi k(\mu\tau + \sigma)) \end{bmatrix} \end{aligned} \quad (38)$$

where $A, B, C, D, a_{Jk}, b_{Jk}, c_k, d_k$ are arbitrary constants,

$$\alpha_{Jk} \equiv \frac{-\omega_{Jk}^2 + 4\pi^2 k^2}{(-4\pi^2 k(1 + \mu^2) + 4\pi\omega_{Jk}\mu)}, \quad (39)$$

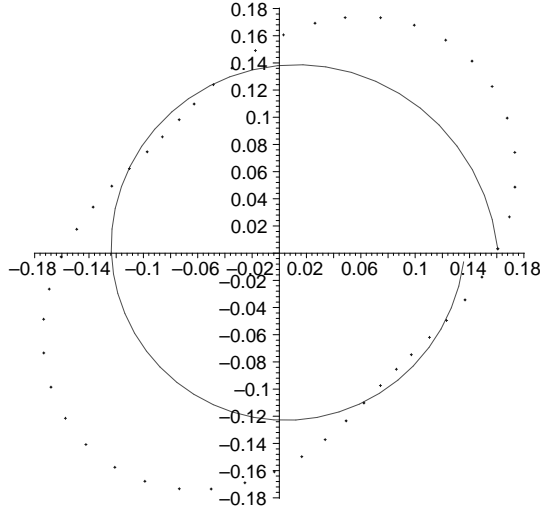


Figure 1. The $k = 2$ motion for ω_{12}

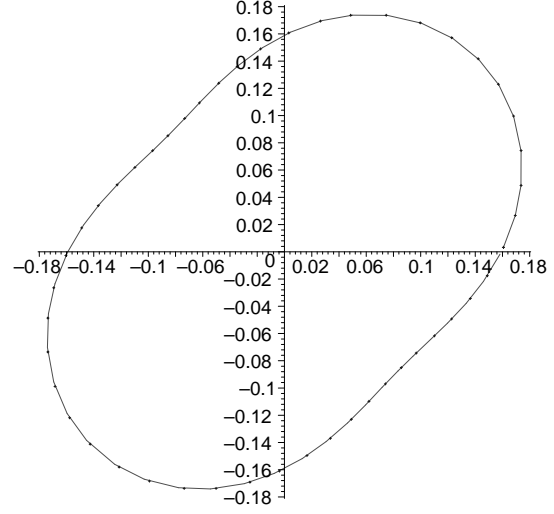


Figure 2. The $k = 2$ motion for ω_{22}

$$\begin{aligned}\omega_0 &= 2\pi\sqrt{1+3\mu^2}, \\ \omega_{11} &= -2\pi(\mu - \sqrt{2(1+\mu^2)}), \\ \omega_{21} &= -2\pi(\mu + \sqrt{2(1+\mu^2)}),\end{aligned}$$

and, for each $k \geq 2$ and $J \in [1, 4]$, ω_{Jk} are the 4 solutions to the following equation in x_k

$$(2\pi\mu k - x_k)[x_k^3 + 2k\pi\mu x_k^2 - 4\pi^2(3\mu^2 + k^2 + 1)x_k + 8k\pi^3\mu(\mu^2 - k^2 + 3)] = 0. \quad (40)$$

Solution, (38), satisfies the following restriction on the centre of mass:

$$\int_0^1 \mathbf{r} d\sigma = 0.$$

When $k = 0$ the term linear in τ can be interpreted as changes to the angular velocity of the ring. The \mathbf{U}_3 motion, governed by ζ , is a string in static equilibrium and in the special case of $k = 2$ can be interpreted as a fixed rigid body rotation of the ring about a bisecting axis.

Apart from the zeroth and first modes there are 4 characteristic frequencies associated with each mode shape governed by k . For $k = 2$, figures (1) to (4) show the mode shape at $\tau = 0$ (dotted line) and the motion of the material point $\sigma = 0$ (continuous line) over a time period just short of one period for a stretch of $\nu_0 = 1.01$. Figure (2) clearly shows an axially flowing ‘Healey’ loop [10]. Also of interest is the motion corresponding to the lowest frequency which nearly approximates rigid body rotation (exact rotation is characterized by circular motion). For higher stretch and modes the picture is essentially the same.

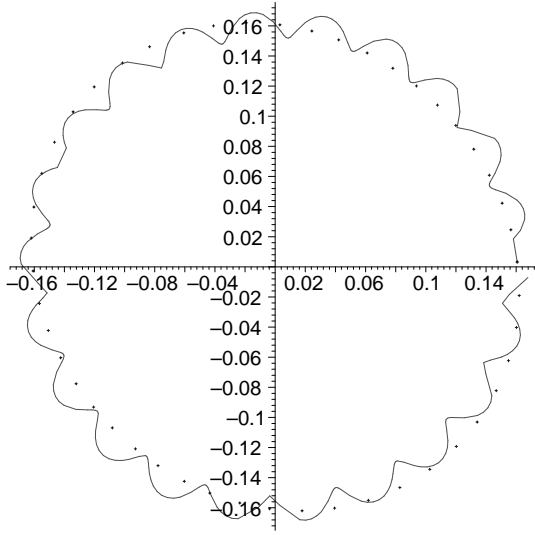


Figure 3. The $k = 2$ motion for ω_{32}

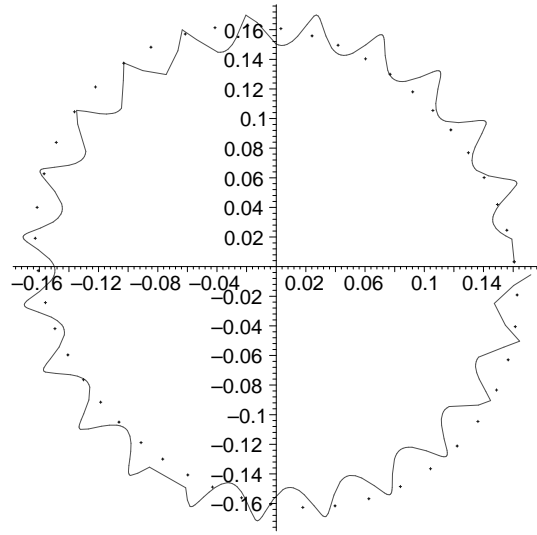


Figure 4. The $k = 2$ motion for ω_{42}

7. Motion of the ring in a plane monochromatic gravitational wave

In this section, using both Fermi coordinates and a related system (*plane* normal coordinates), the relative acceleration associated with a monochromatic plane gravitational wave is derived on a volume of space surrounding a line of observers. The wave frequencies that parametrically excite the ring of section (6) are determined and non geodesic motion due to 'spin' coupling is demonstrated.

Writing

$$g \equiv g_M + h \quad (41)$$

where, with $e^0 \equiv c dt$, $e^1 \equiv dx$, $e^2 \equiv dy$, $e^3 \equiv dz$, g_M is the Minkowski metric

$$g_M \equiv -e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3, \quad (42)$$

c is the speed of light and h is defined

$$h \equiv \epsilon \kappa (2xy + x^2 - y^2) \sin \left(\frac{2\pi(z - ct)}{\lambda} \right) (e^0 - e^3) \otimes (e^0 - e^3). \quad (43)$$

The metric g is an *exact* vacuum solution of Einstein's equations for arbitrary dimensionless constant ϵ , constant κ with dimensions L^{-2} and represents a plane monochromatic gravitational wave [11],[1]. Now the frame $\{e^a\}$ defines the metric dual frame $\{X_a\}$ which coincides with a Fermi frame $\{\mathbf{W}_a = X_a|_C, a = [1, 3]\}$ for an observer defined by

$$\tau \mapsto C(\tau) \equiv (t = \tau, x = 0, y = 0, z = 0)$$

so that

$$\dot{C} \equiv \partial_\tau = \partial_t$$

and the tidal force at a position $\mathbf{r} \equiv r^a W_a$ relative to C is given by

$$\text{Riem}(\dot{C}, \mathbf{r}, \dot{C}, -) = \epsilon c^2 \kappa \sin\left(\frac{-2\pi c\tau}{\lambda}\right) [(r^1 + r^2)\mathbf{W}_1 + (r^1 - r^2)\mathbf{W}_2]. \quad (44)$$

valid for small $r^3 \ll \lambda$. The tidal force can be interpreted [11] by its effects in a tangent plane, W , normal to the propagation direction $Y \equiv \frac{1}{c}\partial_t + \partial_z$, lying on the intersection of the rest space of a geodesic observer and the level hypersurfaces of $ct - z$. For the observer C the plane W is spanned by \mathbf{W}_1 and \mathbf{W}_2 at $x = y = z = 0$, and can be interpreted as the 2-dimensional wavefront associated with C . For short wave lengths, $r^3 > \lambda$ the tidal force (44) is only reliable when restricted to W since it is only accurate to first order in distance from the observer. To overcome this, consider a one parameter family of observers given by the mapping, S defined by

$$(\tau, v) \mapsto S(\tau, v) \equiv (t = \tau, x = 0, y = 0, z = v)$$

each with 2-dimensional wavefront W^v spanned by ∂_x, ∂_y on $S(\tau, v)$. In this way it is possible to discuss relative accelerations induced by the wave in a volume of space defined by a sequence of tangent planes W^v by

$$\text{Riem}(\dot{S}, \mathbf{r}, \dot{S}, -) = \epsilon c^2 \kappa \sin\left(\frac{2\pi(v - c\tau)}{\lambda}\right) [(r^1 + r^2)\mathbf{W}_1(v) + (r^1 - r^2)\mathbf{W}_2(v)]. \quad (45)$$

where $\dot{S} \equiv \partial_t$ on $S(\tau, v)$ and the frame basis, $\{\mathbf{W}_1, \mathbf{W}_2\}$ depends both on τ and v . If, as is the case here, $S(\tau, v)$ is aligned along a spacelike geodesic ($\nabla_{\partial_v} \partial_v = 0$) then it is possible to associate the vector $r^3 \mathbf{W}_3$ via the exponential map at time τ to an observer in S by $S(\tau, v) = \exp_\tau(r^3 \mathbf{W}_3)$ and hence $v = r^3$.

To put all this in context, consider the following coordinate systems: Normal (or Gaussian) coordinates, which are 4 coordinates defined in a tangent space at an event in spacetime [6]; Fermi normal coordinates which are 3 coordinates defined at a series of one parameter volume tangent spaces; and, for want of a better name, *Plane* normal coordinates which are 2 coordinates defined at a series of two parameter plane tangent spaces representing a string of observers. The latter is what we have here. We might term these (0, 4), (1, 3) and (2, 2) normal coordinate systems respectively. So, for example, (4, 0) normal coordinates are ordinary coordinates and (3, 1) normal coordinates presumably have application for world volume corresponding to a surface of observers.

We now inspect the force on the unperturbed spinning ring (33) with observer C located at the circle centre and assume $\epsilon \ll 1$ so that a force \mathbf{f} can be equated with a perturbation to the static equilibrium solution (33) so that (36) can be applied. On the ring

$$\begin{aligned} r^1 &= r_0 \cos \theta, \\ r^2 &= r_0 \sin \theta, \\ r^3 &= 0, \end{aligned} \quad (46)$$

where $r_0 \equiv 1/(2\pi(1 - \mu^2))$, $\theta \equiv 2\pi(\sigma + \mu\tau)$ and the Fermi basis $\mathbf{W}_a, a \in [1, 3]$ replaces

the Cartesian basis given in terms of the $\{\mathbf{U}_a\}$ basis by inverting (31)

$$\begin{bmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \\ \mathbf{W}_3 \end{bmatrix} \equiv \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \mathbf{U}_3 \end{bmatrix}$$

Substituting into (44) gives the rhs of (36):

$$\begin{aligned} \mathbf{f} &= r_0 c^2 \kappa \sin \left(\frac{-2\pi c \tau}{\lambda} \right) [(\cos \theta + \sin \theta) \mathbf{W}_1 + (\cos \theta - \sin \theta) \mathbf{W}_2] \\ &= \frac{1}{2} r_0 c^2 \kappa \left\{ [\sin(\phi_+) - \sin(\phi_-) - \cos(\phi_+) + \cos(\phi_-)] \mathbf{U}_1 \right. \\ &\quad \left. + [\sin(\phi_+) - \sin(\phi_-) + \cos(\phi_+) - \cos(\phi_-)] \mathbf{U}_2 \right\} \end{aligned}$$

where

$$\phi_{\pm} \equiv 2\theta \pm \frac{2\pi c \tau}{\lambda} \equiv 4\pi\sigma + (4\pi\mu \pm 2\pi c/\lambda)\tau$$

and light speed, $c \equiv c/c_s$, is a dimensionless constant. We expect resonant effects to take place when the forcing frequencies match the resonant frequencies

$$\omega_{Jk}\tau + 2\pi k\sigma = \phi_{\pm} \equiv (4\pi\mu \pm 2\pi c/\lambda)\tau + 4\pi\sigma$$

This occurs when $k = 2$ and

$$\omega_{J2} = 4\pi\mu \pm 2\pi c/\lambda. \quad (47)$$

Substituting $k = 2$ and (47) into (40) gives a cubic equation for the angular frequency, μ . For example, consider the 'Healey' solution to (40) for $k = 2$ and using the label $J = 2$

$$\omega_{22} = 4\pi\mu.$$

This implies that resonance occurs as the frequency, $f \equiv c/\lambda$, of the gravitational wave approaches zero. Recall that a Healey loop is in static equilibrium and excitation of ω_{22} would result in a gradual distortion of the ring. However, a wave with $f = 0$ is not a wave at all so this result may be discounted. The remaining frequencies are dependent on $\mu \equiv \sqrt{1 - 1/\nu}$. Consider a particular strain, $\nu = 1.01$ implying $\mu = 0.0995$ and solutions to (40) of

$$\omega_{12} = -0.2458, \quad \omega_{22} = 1.2504, \quad \omega_{32} = 13.6067, \quad \omega_{42} = -14.6113.$$

Then corresponding wave frequencies f_J associated with resonance in this example are

$$f_1 = \pm 0.238, \quad f_2 = 0, \quad f_3 = \pm 1.967, \quad f_4 = \pm 2.524.$$

It is not clear which mode of vibration would be easiest to detect.

Assume that there exists a ring corresponding to the above example. The question arises as to how the ring could be altered without effecting the resonant frequencies, f_J . Since f_J are only dependent on time which is measured in units of characteristic time L_0/c_s , there is a class of rings with equal characteristic time which will resonate with a particular wave of given frequency. For example one may keep the characteristic time

constant by doubling length L_0 and also doubling $c_s = \sqrt{E/\rho_0}$ by halving the cross sectional diameter of the string, since the mass per unit length ρ_0 is proportional to area, A . Hence, there is no theoretical limit to the size of the ring. However, practical considerations may decree that the ring may be of comparable size to the wavelength, $L_0 \approx \lambda$ and it is desirable to know how such a detector may respond when not correctly aligned in the wave front. To do this (45) is used in place of (44) and the extreme example where the normal to the ring is perpendicular to the propagation direction is now discussed. With the ring lying in the xz plane

$$\begin{bmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \\ \mathbf{W}_3 \end{bmatrix} \equiv \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ 0 & 0 & -1 \\ \sin \theta & \cos \theta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \mathbf{U}_3 \end{bmatrix} \quad (48)$$

and

$$\begin{aligned} r^1 &= r_0 \cos \theta, \\ r^2 &= 0, \\ r^3 &= v = r_0 \sin \theta. \end{aligned} \quad (49)$$

As with (46) we assume that the ring configuration (33), when written in the Fermi frame is a solution to Cosserat's equations to $O(\epsilon)$, even for short wavelength λ . Substitution of (48) and (49) into the tidal tensor (45) gives

$$\begin{aligned} \text{Riem}(\dot{S}, \mathbf{R}, \dot{S}, -) &= \epsilon c^2 \kappa \sin \left(\frac{2\pi(r_0 \sin \theta - c\tau)}{\lambda} \right) r_0 \cos \theta \mathbf{W}_1 \\ &= \epsilon r_0 c^2 \kappa \sin \left(\frac{2\pi(r_0 \sin \theta - c\tau)}{\lambda} \right) \cos \theta (\cos \theta \mathbf{U}_1 - \sin \theta \mathbf{U}_2). \end{aligned}$$

The case $r_0 \rightarrow 0$ results in terms cosine or sine of $(2\theta \pm 2\pi f\tau)$ which we can use to extract from $\text{Riem}(\dot{S}, \mathbf{R}, \dot{S}, -)$ coefficients of similar terms when $L_0 \approx 2\pi r_0 \in [0, \lambda]$ by inspecting the change in the following integrals

$$\begin{aligned} &\int_0^1 \int_0^b \text{Riem}(\dot{S}, \mathbf{R}, \dot{S}, -) \sin(2\theta + 2\pi f\tau) d\tau d\sigma, \\ &\int_0^1 \int_0^b \text{Riem}(\dot{S}, \mathbf{R}, \dot{S}, -) \cos(2\theta + 2\pi f\tau) d\tau d\sigma, \end{aligned}$$

where $b \equiv 1/(2\mu + f)$ and $f \equiv \pm c/\lambda$. Numerical evaluation indicates little change for L_0 in the range given, implying that resonant effects are not destroyed when the ring is not aligned perfectly in the wave front.

To examine the motion of the centre of mass (CM), note that by the linearity of the tidal tensor and the symmetry of the ring that the acceleration of the CM vanishes

$$\int_0^1 \text{Riem}(\dot{C}, \mathbf{R}, \dot{C}, -) d\sigma = 0.$$

However the term

$$\epsilon \ddot{\mathbf{R}}_{CM} \equiv 2 \int_0^1 \text{Riem}(\dot{C}, \mathbf{R}, \dot{\mathbf{R}}, -) d\sigma.$$

does not vanish in general. Using the ring configuration (48) and (49) implying

$$\dot{\mathbf{R}} = 2\pi\mu r_0(-\sin\theta\mathbf{W}_1 + \cos\theta\mathbf{W}_3).$$

Specializing to long wavelengths ($r^3 \ll \lambda$), gives

$$\ddot{\mathbf{R}}_{CM} = 2\pi r_0^2 \kappa c \mu \sin(2\pi f\tau)(\mathbf{W}_1 + \mathbf{W}_2)$$

and hence the centre of mass \mathbf{R}_{CM} will oscillate in simple harmonic motion about the geodesic observer with amplitude proportional to the spin frequency, μ . Finally, since $\mathbf{n} = (1 - 1/\nu)\mathbf{r}' \equiv \mu^2\mathbf{r}'$ it is easily verified that $\int_0^1 \text{Riem}(\mathbf{r}, \mathbf{r}', \mathbf{n}, -) d\sigma$, vanishes in both orientations and thus does not contribute to the CM motion.

8. Conclusion

This paper has derived a general relativistic string, its reduction in the special case of flat spacetime, its form in a local Fermi frame and, for non relativistic speeds a formulation as a Cosserat string plus three gravitational terms. By perturbative analysis the normal modes associated with a Cosserat string arranged as an axially flowing ring were investigated. This allowed analysis of the effects associated with a ring in a plane monochromatic gravitational wave, in particular the excitation of the ring's natural frequencies and the oscillation of its centre of mass about geodesic motion.

There remain, of course, significant practical obstacles to the construction of a lasso antenna, not least how to convert string motion to data with sufficient sensitivity to extract those resonances associated with gravitation waves from more dominant Newtonian effects. Piezoelectric material or optical fibre may have properties that can be exploited in this respect.

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